Charles Hermite Strikes Again

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Most physicists think of quantum mechanics when they think of Hermite polynomials. Several of Tom Armstrong's students have come to know these polynomials only too well, but for another application. Under Armstrong's direction, my dissertation involved the construction of a Vlasov plasma simulation code that used an expansion of the velocity dependence of the distribution function **f** in Hermite polynomials. The expansion takes the form

$$\mathbf{f}(\mathbf{x},\mathbf{v},t) = e^{-\mathbf{v}\cdot\mathbf{v}/2} \sum_{i,j,k=0}^{\infty} f_{i,j,k}(\mathbf{x},t)h_i(v_r)h_j(v_\theta)h_k(v_z)$$

where $h_j(v)$ is the Hermite polynomial (modified slightly with an extra factor of 2 in the weight function). The transformed Vlasov equation reduces to a series of interrelated finite difference equations for the coefficients of that expansion. (The details can be found in my thesis or in J. Comp. Phys. **16**, 49 (1974).) These coefficients can be advanced in time and then re-summed to find the space and velocity dependence of **f**.

Given the tendency of a velocity distribution to evolve towards a Maxwellian, the Hermite polynomial is a clever choice for this application. A perfect Maxwellian can be represented by a single term–the first coefficient $f_{0,0,0}$ is proportional to the density. The next coefficients are proportional to the current and is thus related to the drift of the Maxwellian. For example, $f_{0,0,1}$ is proportional to the current in the z-direction. Many plasmas can, therefore, be well represented by only a few terms of the expansion. Higher moments of **f** require combinations of more expansion coefficients–reflecting the need for more terms as **f** becomes more unlike the Maxwellian weight function. I wish that I could take credit for the idea, but my work had a slightly different focus: I generalized and extended to cylindrical geometry these transform techniques that Armstrong used while he was a student at Iowa.

My choice of Tom Armstrong as a thesis advisor was not based on his ability to do Hermite expansions. When I started thinking about choosing an adviser, the hallway consensus was that Armstrong, though new to the business, would probably do a very good job. A few, who had taken courses from him, also voiced the opinion that one might have to work hard to "get out". They were near the mark on all counts. Working with Tom was pleasant but challenging–coming up with concepts that he had not already considered and used or discarded required considerable effort. However, long before "no pain, no gain" became a fashionable thing to say, I stumbled onto a "new" scaling scheme that I became convinced would allow us to run simulations with much larger time steps. Unfortunately, my idea had two problems. The *second* problem should have made me suspicious. The scheme was too good; it allowed any time step. Later, it turned out that as Δt increased, the scaling parameter ω_{pe} correspondingly decreased, leaving the product, a measure of the real time by which the simulation has progressed, unchanged. Certainly no gain there; now for the pain. This second problem was a result of the *first* problem; I had made a simple error in deriving the new scheme. I still have the letter Tom sent from Johns Hopkins gently "deflating" this scheme but encouraging me to try again.

A remarkable quality about Tom Armstrong is his infectious optimism and enthusiasm. I had embarked on a rather ambitious simulation code that would demand far greater computer capacity than was available at KU in 1971. The plan was to visit major laboratories during the summers to obtain the needed time. To this end, Tom secured an invitation to Los Alamos National Laboratory (then LASL) for a summer and again the following spring for a short visit that coincided with a magnetic fusion conference. This trip followed a classic minimal-dollar-burn "flight plan". We left Lawrence about 2 p.m. the day before the meeting, split the driving through the night in Tom's station wagon, and arrived just before dawn in Los Alamos. After sleeping a couple more hours in the car, we checked into a motel, showered, and went to the meeting. During the course of the meeting and later in the evening, we started running the code on the then-mighty CDC 6600s at Los Alamos. At the end of the meeting, on the third day, we left on a "red-eye" drive back to Lawrence. I could not have asked for more support with my efforts to develop the code.

To further test our resolve, we encountered heavy fog on the return trip and were slowed to a crawl until a large truck passed us. Perhaps the truck driver could see much better than I, but, for whatever reason, he seemed more than willing to clear our path at twice the speed we had been traveling. Though I could see only a short distance through the fog, I could see the lights on the top of the truck from some distance so I followed these lights. One of my favorite memories is the image in the rear view mirror as Tom's head popped up to see what was happening. Of course, all he could see was this smiling graduate student driving like crazy through the fog. I am sure he was convinced (or at least hoped) he was dreaming; without a word he rolled over and went back to sleep.

The rest of my dissertation work was not as memorable, though it did involve several more "exciting" trips to computer meccas. I finally got the dissertation assembled and emerged from KU with both a shiny Ph.D and shiny second Lieutenant's bars (Uncle Sam wanted warm bodies during my graduate school days). I was somewhat dismayed by the discovery that I was one of the few people in the "real" world who knew–by memory–the first five Hermite polynomials and their recurrence relations! My colleagues in the US

Army Ordinance Corps at Aberdeen were certainly impressed! Given my short attention span, one would not be surprised that, after the short stint in the Army, I was anxious to move on to new things that did not involve Hermite polynomials. Benefiting from the summer trip with Tom, I landed a job at Los Alamos.

My first task at Los Alamos was to generate an inhomogeneous Vlasov equilibrium. We needed a magnetized equilibrium with gradients strong enough to quickly excite plasma instabilities, so that we could simulate their nonlinear behavior with particle-in-cell PIC methods. "Everyone" knows that you just have to choose a **f** that is a function of the constants of the motion to find an equilibrium. The trouble is that one has to be *very* selective in the choice of that functional form to obtain a useful equilibrium. Fortunately, for those who know the first five Hermite polynomials, there is a trick! If we expand the Vlasov equation in Hermite polynomials in the v_θ coordinate, the transformed equation can be written in a form that allows one to choose the spatial dependence of the low-order macroscopic moments of the equilibrium and construct all higher moments in v_θ. The transformed equation takes the form

$$f_j(x) = \frac{-1}{\sqrt{j}B_z(x)}\left(\frac{d}{dx}f_{j-1}(x) - E_x(x)f_{j-1}(x)\right),$$

which is used as a recurrence relation for the expansion coefficients. In principle, this relation allows the construction of all higher moments. The method does not work perfectly because of the increasing order of numerical derivatives required for high-order coefficients. The method does work much better, however, than trying to guess a functional form that gives low-order moments that are "close enough" to observation to be useful. Since our purpose was to build an equilibrium to initialize a PIC code, we took advantage of the fact that computationally tractable numbers of particles per cell cannot resolve these subtleties in more than a couple of moments beyond the temperature. The method does work very well–we extended and obfuscated this technique in a 1976 Physics of Fluids paper.

I became firmly committed to the promise of fusion energy during my years at Los Alamos, where I continued to write and apply computer models. I finally did get away from Hermite polynomials. My wanderlust also continued–leading me to spent a year at the Plasma Fusion Center at MIT during the heady days of pellet injection and Lawson breakeven in Alcator. On leaving MIT, an opportunity arose to work on inertial confinement fusion (ICF) at Lawrence Livermore National Laboratory (LLNL) so we moved to Livermore rather than back to Los Alamos.

LLNL has proven to be a wonderful environment in which to do a wide variety of research, ranging from Direct Implicit EM PIC to magnetic reconnection in the magnetosphere to plasma-mesh interactions pertinent to ion sources needed for heavy-ion-driven ICF. Recently I have gotten involved in another aspect of this program, namely the transport of high-current, space-charge-dominated heavy ion beams in recirculating induction linacs. We are optimistic about the program, and, with recent favorable reviews by a committee from the National Academy of Sciences and the Fusion Power Advisory Committee, we hope the program will grow. Though the present program is modest by hot-fusion standards ($\sim$ \$10 million, 1991), we are pleased with the time derivatives of the funding.

Another pleasant attribute of LLNL is the close association with the University of California at Davis. A program exists that allows graduate students to work on thesis topics at LLNL as they pursue advanced degrees sanctioned by UC Davis. I am now advising several of these students. One student has just written a code based on a new type of electromagnetic field model we developed. We needed a nontrivial test case with strong gradients to see if we could maintain stable equilibria without dissipation. Fortunately, for those who know..., there is a trick! As you can probably guess, there now exists another person who knows the first five Hermite polynomials by heart.