An X-type Reconnection Field Model

Patrick R. Briggs

Department of Physics, The Citadel

1. Introduction

The story of this article begins in the Summer of 1982 during a postdoctoral stay at KU. Frank Kutchko and I were having a series of very enjoyable "arguments" about one of his favorite topics: magnetic reconnection. He defended his favorite field models, while I (playing Devil's Advocate) tried to point out flaws. Our discussions soon focused on the physical *reasonableness* of each model: does it look like a "real" field? When I insisted that any field model should at least satisfy $\nabla \cdot \mathbf{B} = 0$ globally, he pointed out that while Nature knows how to construct such a field, it might not have a simple analytical formula. He was, of course, correct; nevertheless, the possibility of a relatively simple formula for an X-type field topology was a compelling idea.

The search for that simple formula became part amusement, part obsession over the next couple of years. It's the sort of problem that sits in the back of a file cabinet for several months while your "real" work gets done, and then gets pulled out and chewed on over a free weekend. Several false leads and heroic failures eventually gave way to a solution that at least had the look-and-feel of a real X-type reconnection field.

Now since this volume is not a formal journal, we don't have to follow one of the cardinal rules of scholarly articles:

The author's result must appear to have been derived from first principles by a dispassionate and unerringly consistent application of Aristotelian logic.

Of course, nobody actually *works* that way, but papers are always *written* that way. In this article, the result will be derived twice: first, showing the author's actual thought processes (if any) as he worked toward a solution; second, in polished form suggesting that the solution leapt Athena-like from the author's head.

2. The Problem

We start with the magnetic field due to a current sheet. If the sheet can be considered infinitely wide, the field has the following asymptotic behavior: it is constant and parallel to the sheet at great distances, and the field direction changes as you pass through the sheet [Figure 1].
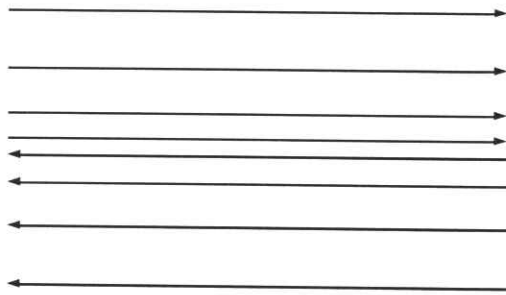
Figure 1. Magnetic field due to a current sheet

Now imagine the field structure that arises if the field lines reconnect: lines near the sheet have been torn like rubber bands and rejoined across the sheet, forming the characteristic X-type configuration [Figure 2]. The task is to write a formula for a two-dimensional X-type field with the right qualitative shape and no divergence.
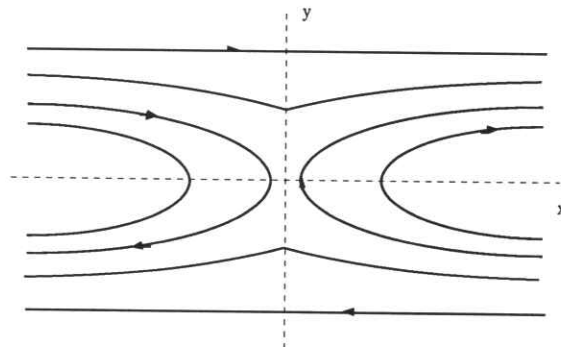
Figure 2. An X-type reconnection field

31

3.   A Solution

First we look at the behavior of the field far from the reconnection; it should look like an undisturbed sheet field [Figures 3,4].
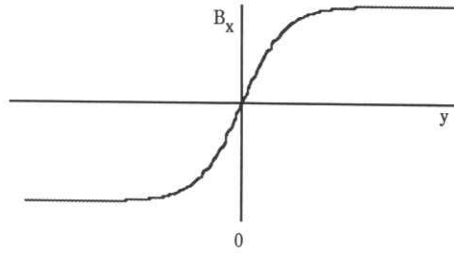

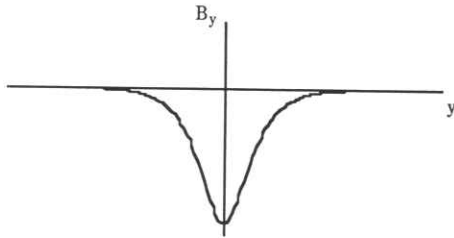Figure 3. B_x as a function of y far from the origin

Figure 4. B_y as a function of y $(x < 0)$

Along the y-axis, the field has no y-component; B_x in fact has the same qualitative form as in Figure 3. A look at Figure 3 suggests a formula for B_x like $B_x(y) \sim \tanh(ay)$, where **a** is a constant. The dependence on x is a bit more subtle:

Figure 2 seems to say that B_x will show a drop in magnitude near $x = 0$, with the effect decreasing as one gets farther from the x-axis. Additionally, B_x must be zero along the x-axis itself.

It seemed to me that if a set of compatible field equations were to be found, they would need to be built from the hyperbolic trigonometric functions. Now Figure 4 above looks like an (inverted) hyperbolic secant function. Checking my trusty (and completely dog-eared) copy of Schaum's *Mathematical Handbook*, I found that

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

and so taking the divergence of a field that had tanh x and sech x would yield the same sort of functions.

The field that I settled on had the following form:

$$B_x = B_0 \tanh(ay) \cdot \left[1 - \operatorname{sech}(ax)\operatorname{sech}(ay) \right] \qquad (1)$$

$$B_y = B_0 \tanh(ax)\operatorname{sech}(ax)\operatorname{sech}(ay) \qquad (2)$$

This field is divergenceless; while this result may be "left to the reader," here it is anyway:

$$
\begin{aligned}
\nabla \cdot B &= \frac{\partial}{\partial x}B_x + \frac{\partial}{\partial y}B_y \\
&= B_0 \tanh(ay)\operatorname{sech}(ax)\tanh(ax)\operatorname{sech}(ay) - B_0\tanh(ax)\operatorname{sech}(ax)\operatorname{sech}(ay)\tanh(ay) \\
&= 0
\end{aligned}
$$

Incidentally, the first term in equation (1) is just the field of the current sheet; if we take that term out of the equations, we have the mathematical model of an "O-type" region: the field due to a current beam.

33

4.   The Exposition

First we construct a current sheet with current density of the form

$$\vec{J}_0 = J_0 \, \text{sech}^2 ay \, \hat{k}$$

where J_0 and a are constants. Then we introduce an "interruption current" of the form

$$\vec{J}_1 = J_1 \, \text{sechax} \, \text{sechay} \left(\tanh^2 ax + \tanh^2 ay - 1 \right) \hat{k}.$$

Then taking our current density as the sum of the sheet current and the interruption current, we can calculate the magnetic field from

$$\vec{B} = \int_V \vec{\nabla} \times \frac{\vec{J}}{|\vec{r}|} d^3 V.$$

With suitable choice of constants, this generates the magnetic field given by (1) and (2) above.

q.e.d.